A REMARK ON UNCONDITIONAL BASIC SEQUENCES IN L_p $(1 < p < \infty)$[†]

BY

G. SCHECHTMAN

ABSTRACT

We prove that every separable $\mathcal{L}_p$ space $(1 < p < \infty)$ with an unconditional basis is isomorphic to a complemented subspace of L_p which is spanned by a block basis of the Haar system.

1. Introduction

In [1] the authors ask what are the isomorphic types of complemented closed linear spans of block bases of the Haar system in L_p $(= L_p(0,1))$. In particular, is every $\mathcal{L}_p$ space $(1 < p < \infty)$ with an unconditional basis isomorphic to such a space?

In this paper we show by using essentially known techniques that the answer to this question is affirmative.

2.

PROPOSITION 1. *Let $\{x_i\}_{i=1}^{\infty}$ be an unconditional basic sequence in L_p $(1 < p < \infty)$. Then $\{x_i\}_{i=1}^{\infty}$ is equivalent to a block basis of the Haar system.*

PROOF. It is a well-known fact that there exists a $K > 0$ such that for any choice of scalars $\{a_i\}_{i=1}^{\infty}$,

$$(1) \qquad K^{-1}\left\|\sum_{i=1}^{\infty} a_i x_i\right\| \le \left(\int_0^1 \left(\sum_{i=1}^{\infty} a_i^2 x_i^2(t)\right)^{p/2} dt\right)^{1/p} \le K\left\|\sum_{i=1}^{\infty} a_i x_i\right\|.$$

(This inequality is proved by integrating against the Rademacher functions.)

Thus, if $\{y_i\}_{i=1}^{\infty}$ is another unconditional basic sequence such that $|y_i| \equiv |x_i|$, $i = 1, 2, \cdots$, then $\{y_i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$. Now let $\{\phi_i\}_{i=1}^{\infty}$ be the Haar system, i.e.,

[†]This is part of the author's Ph.D. thesis written at the Hebrew University of Jerusalem under the supervision of Professor J. Lindenstrauss.

Received April 1, 1974

$$\phi_{2^n+k} = \chi_{(k2^{-n},(k+1/2)2^{-n})} - \chi_{((k+1/2)2^{-n},(k+1)2^{-n})}$$

for $n = 0,1,2,\cdots,k = 0,1,\cdots,2^n - 1$, and $\phi_0 \equiv 1$ (where χ_A denote the characteristic function of A).

By the usual stability theorems we may assume without loss of generality that there exists a strictly increasing sequence of integers $\{m_n\}_{n=1}^{\infty}$ such that

$$x_n = \sum_{k=0}^{2^{m_n}-1} a_{2^{m_n}+k}\,\chi_{(k2^{-m_n},(k+1)2^{-m_n})} = \sum_{k=0}^{2^{m_n}-1} a_{2^{m_n}+k}\,|\,\phi_{2^{m_n}+k}\,| \qquad n = 1,2,\cdots.$$

Put, for $n = 1,2,\cdots$,

$$y_n = \sum_{k=0}^{2^{m_n}-1} a_{2^{m_n}+k}\,\phi_{2^{m_n}+k}.$$

Then, $|y_n| \equiv |x_n|$, $n = 1,2,\cdots$, and thus, as remarked above, $\{y_n\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$.

COROLLARY 1. *Every unconditional basis of L_p $(1 < p < \infty)$ is reproducible.*

Let us recall first that a basis $\{x_i\}_{i=1}^{\infty}$ in a Banach space X is called *reproducible* if for every embedding of X in a Banach space Y with a basis $\{y_i\}_{i=1}^{\infty}$, $\{x_i\}_{i=1}^{\infty}$ is equivalent to a block basis of $\{y_i\}_{i=1}^{\infty}$.

PROOF OF THE COROLLARY. By [2] the Haar basis is reproducible. The rest is a simple consequence of Proposition 1.

We shall denote by $L_p(l_2)$ the Banach space of all sequences $\{f_1, f_2, \cdots\}$ of functions on [0,1] such that:

$$\| \{f_i\}_{i=1}^{\infty} \|_{L_p(l_2)} = \left(\int \left(\sum_{i=1}^{\infty} f_i^2(t) \right)^{p/2} dt \right)^{1/p} < \infty. \tag{2}$$

LEMMA 1. ([3, Vol. II, Lemma 2.10, p. 224]). *Let T: $L_p \to L_p$ be a bounded operator. The operator $\tilde{T}$: $L_p(l_2) \to L_p(l_2)$ defined by*

$$\tilde{T}(f_1, f_2, \cdots) = (Tf_1, Tf_2, \cdots)$$

is a bounded operator in $L_p(l_2)$. (In fact $\|\tilde{T}\| = \|T\|$.)

LEMMA 2. *Let $\{x_i\}_{i=1}^{\infty}$ be an unconditional basic sequence in L_p $(p > 2)$. Then there exists a $K > 0$ such that for all scalars $\{a_{i,j}\}_{i,j=1}^{\infty}$ (with only finitely many $\neq 0$):*

$$\begin{aligned} &K^{-1} \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i,j}^2 \right)^{1/2} x_i \right\|_{L_p} \\ &\le \left(\int \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{i,j} x_i(s) \right)^2 \right)^{p/2} ds \right)^{1/p} \le K \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i,j}^2 \right)^{1/2} x_i \right\|_{L_p}. \end{aligned} \tag{3}$$

PROOF. In what follows the sign $\approx$ will denote the existence of inequalities in both directions with constants which do not depend on the scalars $\{a_{i,j}\}_{i,j=1}^{\infty}$. By

$\{r_i\}_{i=1}^{\infty}$ we shall denote the Rademacher functions, i.e.,

$$r_k(t)=\begin{cases} 1 & \text{if} \quad j/2^k \leqq t<(j+1)/2^k, j \text{ even} \\ -1 & \text{if} \quad j/2^k \leqq t<(j+1)/2^k, j \text{ odd}. \end{cases}$$

Notice first that by Khinchine's inequality and (1),

$$\begin{aligned} &\left(\int\left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{i,j}x_i(s)\right)^2\right)^{p/2} ds\right)^{1/p} \approx \left(\int\int\left|\sum_{j=1}^{\infty} r_j(t)\sum_{i=1}^{\infty} a_{i,j}x_i(s)\right|^p dt\,ds\right)^{1/p} \\ &= \left(\int\int\left|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)x_i(s)\right|^p ds\,dt\right)^{1/p} \approx \left(\int\int\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)\right)^{p/2} ds\,dt\right)^{1/p}. \end{aligned} \tag{4}$$

Now, using the fact that $x^{p/2}$ is convex function, the fact that the Rademacher functions form an orthonormal system and again (1) we obtain:

$$\begin{aligned} &\left(\int\int\left|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)\right|^{p/2} dt\,ds\right)^{1/p} \\ &\geq \left(\int\left(\int\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)dt\right)^{p/2} ds\right)^{1/p} \\ &= \left(\int\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{i,j}^2\right)x_i^2(s)\right)^{p/2} ds\right)^{1/p} \\ &\approx \left(\int\left|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{i,j}^2\right)^{1/2} x_i(s)\right|^p ds\right)^{1/p} \end{aligned}$$

which proves the left-hand side of (3). On the other hand:

$$\begin{aligned} &\left(\int\int\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)\right)^{p/2} dt\,ds\right)^{1/p} \\ &= \left(\int\left\|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)\right\|_{p/2,t}^{p/2} ds\right)^{1/p} \\ &\leq \left(\int\left(\sum_{i=1}^{\infty}\left\|\left(\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right)^2 x_i^2(s)\right\|_{p/2,t}\right)^{p/2} ds\right)^{1/p} \\ &= \left(\int\left(\sum_{i=1}^{\infty} x_i^2(s)\left\|\sum_{j=1}^{\infty} r_j(t)a_{i,j}\right\|_{p,t}^2\right)^{p/2} ds\right)^{1/p} \\ &\approx \left(\int\left(\sum_{i=1}^{\infty} x_i^2(s)\sum_{j=1}^{\infty} a_{i,j}^2\right)^{p/2} ds\right)^{1/p} \approx \left\|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{i,j}^2\right)^{1/2} x_i\right\|_{L_p}. \end{aligned}$$

(We used the triangle inequality, Khinchine's inequality and (1).) This, together with (4) completes the proof.

LEMMA 3. *Let $\{x_i\}_{i=1}^{\infty}$, be an unconditional basic sequence in $L_p(1 < p < \infty)$, and assume that there exists a projection*

$$P: L_p \xrightarrow{\text{onto}} [x_i]_{i=1}^{\infty} \ ([x_i]_{i=1}^{\infty} = \text{Span}\{x_i\}_{i=1}^{\infty}).$$

For each $1 \leq i < \infty$ put $f^i = (0, \cdots, x_i, \cdots)$ (x_i stands in the i's coordinate). Then $\{f^i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$ and there exists a projection from $L_p(l_2)$ onto $[f^i]_{i=1}^{\infty}$.

PROOF. The fact that $\{f^i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$ follows from (1).

Assume now that $p > 2$. Put $f^{i,j} = (0, \cdots, x_i, \cdots)$, $i, j = 1, 2, \cdots$, (x_i stands in the j's coordinate).

Note that the operator $\tilde{P}$, corresponding to P by Lemma 1, is a projection with range $[f^{i,j}]_{i,j=1}$. The sequence $\{f^i\}_{i=1}^{\infty}$ is a subsequence of $\{f^{i,j}\}_{i,j=1}^{\infty}$ ($f^i = f^{i,i}$, $i = 1, 2, \cdots$) which is an unconditional basic sequence by Lemma 2. This completes the proof of the lemma for the case $p > 2$. For $p < 2$ let $\{y_i\}_{i=1}^{\infty}$ be the sequence in $L_p^* = L_q$ ($1/p + 1/q = 1$) defined by:

$$Pf = \sum_{i=1}^{\infty} y_i(f)x_i \qquad f \in L_p.$$

Then

$$P^*g = \sum_{i=1}^{\infty} x_i(g)y_i \qquad g \in L_q$$

(where P is considered as an operator from L_p to L_p).

Now $q > 2$ and by the first part of the proof there is a projection from $L_q(l_2)$ onto $[g^i]_{i=1}^{\infty}$ where $g^i = (0, \cdots, y_i, \cdots)$.

Denote this projection by Q. By inspecting the first part of the proof one can see that

$$Q(g_1, g_2, \cdots) = (P^*g_1|_{[y_1]}, P^*g_2|_{[y_2]}, \cdots)$$

$$= (x_1(g_1)y_1, x_2(g_2)y_2, \cdots) = \sum_{i=1}^{\infty} f^i(\bar{g})g^i$$

where $\bar{g} = (g_1, g_2, \cdots) \in L_q(l_2)$. (The dual of $L_q(l_2)$ is $L_p(l_2)$ under the pairing $[(f_1, f_2, \cdots)]\ ((g_1, g_2, \cdots)) = \int \Sigma_{i=1}^{\infty} f_i g_i$).

The conjugate projection Q^*: $L_p(l_2) \to L_p(l_2)$ is given by:

$$Q^*(\bar{f}) = \sum_{i=1}^{\infty} g^i(\bar{f})f^i, \qquad \bar{f} = (f_1, f_2, \cdots) \in L_p(l_2).$$

(Again, Q is considered as an operator from $L_q(l_2)$ to $L_q(l_2)$.) Thus $[f^i]_{i=1}^{\infty}$ is complemented in $L_p(l_2)$. The case $p = 2$ is trivial.

THEOREM 1. *Let $\{x_n\}_{n=1}^{\infty}$ be an unconditional basic sequence in $L_p(1 < p < \infty)$ and assume that $[x_n]_{n=1}^{\infty}$ is complemented in L_p. Then $\{x_n\}_{n=1}^{\infty}$ is equivalent to a block basis of the Haar system whose closed span is complemented in L_p.*

PROOF. As in the proof of Proposition 1 we may assume without loss of generality that there exists a strictly increasing sequence of integers $\{m_n\}_{n=1}^{\infty}$ such that:

$$x_n = \sum_{k=0}^{2^{m_n}-1} a_{2^{m_n}+k} \,|\phi_{2^{m_n}+k}|.$$

Put $f^i = (0, \cdots, x_i, \cdots)$, $i = 1, 2, \cdots$, (x_i in the i's coordinate) and,

$$h^{2^{m_n}+k} = (0, \cdots, |\phi_{2^{m_n}+k}|, \cdots)\, n = 1,2,\cdots, k = 0,\cdots,2^{m_n}-1$$

($|\phi_{2^{m_n}+k}|$ stands in the n's coordinate).

Now, $\{f^i\}_{i=1}^{\infty}$ is equivalent to $\{x_i\}_{i=1}^{\infty}$ and is a block basis of $\{h^{2^{m_n}+k}\}$, $n = 1, 2, \cdots, k = 0, \cdots, 2^{m_n} - 1$. By Lemma 3 there is a projection from $[h^{2^{m_n}+k}]$, $n = 1,2,\cdots, k = 0,\cdots,2^{m_n} - 1$ onto $[f^i]_{i=1}^{\infty}$. $\{h^{2^{m_n}+k}\}$, $n = 1,2,\cdots, k = 0,\cdots, 2^{m_n} - 1$ is equivalent to $\{\phi_{2^{m_n}+k}\}$, $n = 1,2,\cdots$, $k = 0,\cdots,2^{m_n} - 1$.

To see this, note that

$$\left\| \sum_{n=1}^{\infty} \sum_{k=0}^{2^{m_n}-1} b_{2^{m_n}+k}\, \phi_{2^{m_n}+k} \right\| \approx \left\| \sum_{n=1}^{\infty} r_n(t) \sum_{k=0}^{2^{m_n}-1} b_{2^{m_n}+k}\, \phi_{2^{m_n}+k} \right\|$$

with constants which do not depend on t and integrate with respect to t.

Note that the correspondence between $\{h^{2^{m_n}+k}\}$ and $\{\phi_{2^{m_n}+k}\}$ takes $\{f^i\}_{i=1}^{\infty}$ onto the $\{y_i\}_{i=1}^{\infty}$ of Proposition 1. Since the Haar basis is unconditional, $[\phi_{2^{m_n}+k}]$ is complemented in L_p and therefore $[y_i]_{i=1}^{\infty}$ is also complemented in L_p.

ACKNOWLEDGEMENT

The author wishes to thank Professor Lindenstrauss for his guidance.

REFERENCES

1. L. B. Gamlen and R. J. Gaudet, *On subsequences of the Haar system in $L_p[0,1]$ $(1 < p < \infty)$*, Israel J. Math. **15** (1973), 404–413.

2. J. Lindenstrauss and A. Pelczynski, *Contribution to the theory of the classical Banach spaces*, J. Functional Analysis **8** (1971).

3. A. Zygmund, *Trigonometric Series*, Cambridge, 1959.

DEPARTMENT OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL